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COUPLING EFFECTS IN BENDING PROBLEMS FOR BEAMS OF A SHAPE MEMORY ALLOY

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Coupling effects of boundary-value problems that arise for shape-memory alloys whose phase compositions depend on the acting stresses and whose elastic moduli change with variation in the fraction of martensite are studied. Algorithms and results of solution of a number of beam bending problems are given. It is established that in coupled problems, the changes in the stress-strain state upon cooling proceed more smoothly than in uncoupled problems. This is due to the propagation of the front of the beginning of the transformation over the cross section. Overload of the outer layers of the beam and unloading of the inner layers of the beam are found to be related to the propagation of the front of completion of the transformation.

The unique mechanical properties of shape-memory alloys [1–3] are described using a micromechanical system of governing equations [4–6]. An analytical method for solution of uncoupled boundary-value problems for alloys whose phase composition does not depend on the acting stresses and whose elastic moduli are constant was proposed in [7, 8]. However, these conditions are not met for widely used shape-memory alloys such as titanium nickelide [9] and copper-based alloys [10, 11]. For these materials, the martensite portion cannot be calculated before solution of the problem, which thus becomes coupled. In the present paper, we study effects of this type of coupling using beam bending problems as an example.

1. Formulation of the Problem. The one-dimensional constitutive equations for shape-memory alloys [4–6] are of the form

$$\varepsilon = \varepsilon^1 + \varepsilon^2, \quad \varepsilon^1 = \sigma/E, \quad E = E_1 q + E_2(1-q); \tag{1.1}$$

$$d\varepsilon^2 = (B + a_0\varepsilon^2)\,dq; \tag{1.2}$$

$$q = \sin\left(\frac{\pi}{2}\,\frac{M_1 + k|\sigma| - T}{M_1 - M_2}\right), \quad B = \frac{2}{3}\,c_0\sigma \tag{1.3}$$

for

$$M_2 + k|\sigma| < T < M_1 + k|\sigma|, \quad k\,\mathrm{sign}(\sigma)\,d\sigma > dT \tag{1.4}$$

and

$$q = \cos\left(\frac{\pi}{2}\,\frac{T - A_1 - k|\sigma|}{A_2 - A_1}\right), \quad B = \frac{2}{3}\,\lambda c_0\sigma(q)|_{\dot q>0} + (1-\lambda)\,\frac{a_0\varepsilon^*}{\exp(a_0 q_1) - 1} \tag{1.5}$$

for

$$A_1 + k|\sigma| < T < A_2 + k|\sigma|, \quad k\,\mathrm{sign}(\sigma)\,d\sigma < dT. \tag{1.6}$$

Here q is the volumetric portion of martensite, ε, ε^1, and ε^2 are the total, elastic, and phase strains (thermal strain and the volume effect of the phase transformation are ignored in bending problems), T is the temperature, σ is the longitudinal normal stress (the transverse tangential stresses in the constitutive

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equations are ignored), M_1, M_2 and A_1, A_2 are the temperatures at the beginning and end of the direct and reverse thermoelastic transformation in stress-free material, c_0, a_0, λ, and k are parameters of the material, E_1 and E_2 are Young's moduli for the martensite and austenite states, (1.4) are the conditions of the direct transformation ($\dot{q} > 0$), (1.6) are the conditions of the reverse transformation ($\dot{q} < 0$), and the quantity $\sigma(q)|_{\dot{q}>0}$ at the point in the reverse-transformation region that corresponds to the fraction of the martensite phase q is the stress at the point of the previous region of direct transformation with the same value of q. Equation (1.2) for the phase strain in each stage of the direct and reverse transformations is integrated subject to the initial condition $\varepsilon^2|_{q=q_1} = \varepsilon^*$, where q_1 and ε^* are the values of q and ε^2 attained at the last point of the previous stage.

The constitutive equations are supplemented with the relation of the hypothesis of plane sections

$$\varepsilon^1 + \varepsilon^2 = \beta(T, x)z \tag{1.7}$$

and the equilibrium equation

$$2\int_0^h b(z)\sigma z\, dz = M. \tag{1.8}$$

Here $\beta(T, x)$ is the curvature of the beam, x and z are the coordinates along the neutral axis and the height of the section, $b(z)$ and h are the width and height of the section, and M is the bending moment. Relations (1.1)–(1.8) are sufficient to find the dependence of the curvature β on the moment M. To determine the deflection W in the case of small deflections, we use the relation

$$\beta = -\frac{\partial^2 W}{\partial x^2} \tag{1.9}$$

and the corresponding boundary conditions.

If we set $k = 0$ in (1.3)–(1.6), the problem becomes uncoupled. In this case, q does not depend on z at a uniform temperature distribution over the section. From (1.1), (1.2), and (1.7) it follows that both phase strains and stresses are proportional to z. Therefore, in uncoupled problems, the section stresses are uniquely determined by the bending moment and do not depend on the temperature and phase composition; for identical moments, the stresses are the same in an elastic beam and a beam of a shape-memory alloy.

2. Stress Relaxation in a Beam with a Fixed Curvature. We assume that a beam in the austenite state is imparted a curvature β, after which it begins to cool in the temperature range of the direct martensitic transformation at $\beta = \text{const}$ (as if the beam is "wound" on a round drum). Because of the direct transformation, phase deformations will propagate toward the acting stress, which relaxes as a result. It is easy to see that, according to (1.4), the direct transformation begins at

$$T = T^0 = kE_2\beta h + M_1. \tag{2.1}$$

At $M_1 < T < T^0$, the beam section consists of an elastic core, $0 < |z| < z_0 = (T - M_1)/(kE_2\beta)$, and a transformation zone, $z_0 < |z| \leqslant h$. To obtain the solution in the transformation zone, the stress σ is expressed in terms of the elastic strain by means of (1.1), which, in turn, is expressed in terms of the curvature and phase strain by means of (1.7):

$$\sigma = E(q)(\beta z - \varepsilon^2). \tag{2.2}$$

Substituting (2.2) into (1.2) gives a differential equation for the phase strain:

$$\frac{d\varepsilon^2}{dq} + \Big(\frac{2}{3}\, c_0 E(q) - a_0\Big)\varepsilon^2 = \frac{2}{3}\, c_0 E(q)\beta z. \tag{2.3}$$

The solution of Eq. (2.3) is of the form

$$\varepsilon^2 = \beta z[1 - f(q) + a_0 f(q)F(q)]. \tag{2.4}$$

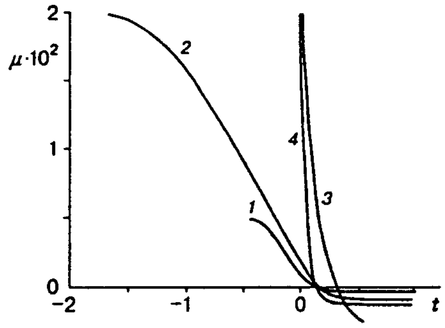

Fig. 1

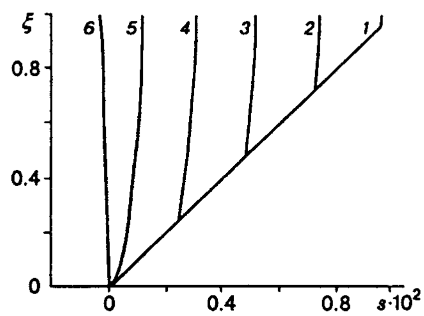

Fig. 2

Here

$$f(q)=\exp\left[\left(a_0-\frac{2}{3}c_0E_2\right)q+\frac{\lambda c_0E_2}{3}q^2\right];\quad \lambda=1-\frac{E_1}{E_2};$$

$$F(q)=\frac{\exp(\gamma^2-a_0)}{\lambda\gamma}[\Phi(\gamma)-\Phi(\gamma-\lambda\gamma q)];\quad \gamma=\left(\frac{c_0E_2}{3\lambda}\right)^{0.5};\quad \Phi(x)=\int_0^x\exp(-\xi^2)\,d\xi.$$

In accordance with (2.2) and (2.4), we have the following relation for σ:

$$\sigma=E_2(1-\lambda q)\beta zf(q)[1-a_0F(q)] \tag{2.5}$$

Here the quantity q is not known in advance in coupled problems. Relations (1.3) and (2.5) are a system of equations that allows one to obtain the dependence of both quantities on the temperature T and the coordinate z. Substituting (2.5) into (1.3) gives a transcendental equation for q, which is solved for chosen values of z, using, e.g., the method of dividing in half.

The solution obtained is valid for $z_0<z<h$ as long as the extreme fiber stress $\sigma(h)$ obtained from this solution satisfies the inequality $T-k|\sigma(h)|\geqslant M_2$, which indicates that the transformation has not been completed yet. Otherwise, this solution is valid only for $z_0<|z|<z_1$, where the coordinate z_1 of the front at the end of the transformation is determined by the formula

$$z_1=\frac{T-M_2}{kE_1\beta f(1)|1-a_0F(1)|}.$$

For $z_1<|z|\leqslant h$, we have $\varepsilon^2=\beta z[1-f(1)+a_0f(1)F(1)]$ and $\sigma=E_1\beta zf(1)[1-a_0F(1)]$. According to the relation obtained, the stress in the zone of completion of the transformation is proportional to z even in the coupled problem, but the proportionality coefficient is different from that in the elastic region.

The bending moment necessary for maintenance of a given curvature $\beta=\text{const}$ is obtained for a rectangular beam from the formula

$$M=2b\Big[\frac{E_2\beta z_0^3}{3}+E_1\beta f(1)(1-a_0F(1))\,\frac{h^3-z_1^3}{3}+\int_{z_0}^{z_1}\sigma(z)z\,dz\Big]. \tag{2.6}$$

The integral in (2.6) is determined numerically.

Figure 1 gives curves of the dimensionless bending moment $\mu=3M/(2bh^2E_2)$ versus the relative temperature $t=(M_1-T)/(M_1-M_2)$ obtained using this formula for titanium nickelide with $E_1=28{,}000$ MPa, $E_2=84{,}000$ MPa, $a_0=0.718$, $k=0.2°\text{C/MPa}$, and $c_0=2.43\cdot10^{-4}$ 1/MPa. Curves 1 and 2 correspond to $\beta h=0.0005$ and 0.002. Evidently, the bending moment necessary for maintenance of a given curvature under cooling in the temperature range of the direct martensitic transformation decreases to zero at a certain temperature $T=T^*$.

One can prove that for an arbitrary continuous dependence of Young's modulus on q, the quantity T^*, if it exists, does not depend on the given curvature, but is determined only by the material constants. Indeed, following (2.3), we represent the solution of the equation for the phase strain as

$$\varepsilon^2 = A(q)\beta z, \tag{2.7}$$

where

$$A(q) = \frac{2}{3} c_0 \int_0^q E(r) \exp\left\{ \int_r^q \left[a_0 - \frac{2}{3} c_0 E(\xi) \right] d\xi \right\} dr. \tag{2.8}$$

The quantity q in (2.7) and (2.8), generally speaking, depends on z via σ according to (1.3).

Since the integrand of the outer integral of (2.8) is continuous and positive, $A(q)$ is a monotonically increasing, continuous function of q. Consequently, if the equation

$$A(q) = 1 \tag{2.9}$$

has a root q^* on the segment $[0, 1]$, this root is single and depends only on the material constants. For $q = q^*$, according to (2.7) and (2.9), the phase strain at each point of the section is equal to the given total strain; hence, the elastic strain and the acting stresses are equal to zero:

$$\varepsilon^1(q^*, z) = 0, \quad \sigma(q^*, z) = 0, \tag{2.10}$$

that is, for $q = q^*$, the stresses completely relax. The corresponding value of T^* can be expressed in terms of q^* using relations (1.3) in which one should set $\sigma = 0$ by virtue of (2.10):

$$t^* = (2/\pi)\arcsin(q^*), \quad T^* = M_1 - (M_1 - M_2)t^*. \tag{2.11}$$

For the given case of linear variation of the elastic modulus (1.1), Eq. (2.9) has the form $F(q) = 1/a_0$ and its solution for material parameters typical of titanium nickelide is $q^* = 0.288$ and $t^* = 0.186$.

It should be noted that, according to (2.8), (2.9), and (2.11), the quantities q^*, t^*, and T^* do not depend on the material parameter k, which determines the effect of the acting stress on q. Consequently, the stress relaxation in an alloy whose phase composition does not depend on the acting stress occurs at the same temperature as in alloys such as titanium nickelide, for which such a dependence exists.

At $T = T^*$, the beam will remain curvilinear with the given constant curvature in the absence of external loads. Further cooling of a beam whose curvature is fixed leads to negative moments M with increasing absolute values (i.e., it is necessary to apply a moment of opposite sign lest the curvature increase). Such behavior is a consequence of oriented transformation [1, 8].

The same problem in an uncoupled formulation is easily solved by the method proposed in [7, 8]. Applying a Laplace transform to the elastic solution $M = E\beta J$ ($J = 2bh^3/3$), replacing the modulus E by the corresponding operator [8], and performing an inverse Laplace transform, we obtain

$$M = \frac{2}{3} bh^3 E\beta \Big[1 - \frac{2c_0 E}{3a_0 - 2c_0 E} \Big(\exp\Big[\Big(a_0 - \frac{2}{3} c_0 E \Big) q \Big] - 1 \Big) \Big].$$

From this solution for $\beta h = 0.002$, curve 3 in Fig 1 is constructed for $E = E_1$ and curve 4 for $E = E_2$. As can be seen, the solutions of the coupled and uncoupled problems differ greatly. The variability of the elastic modulus has a significant effect on T^*. The portion of the martensite phase and the relative temperature at which the moment vanishes are determined from the formulas

$$q^* = \frac{1}{a_0 - 2c_0 E/3} \ln\left(\frac{3a_0}{2c_0 E} \right) \tag{2.12}$$

and (2.11). We have $t^* = 0.3208$ for $E = E_1$ and $t^* = 0.1466$ for $E = E_2$. According to the solutions obtained, in the uncoupled problem, stress relaxation proceeds much more abruptly than in the coupled problem, where deceleration occurs as a result of motion of the transformation front.

Figure 2 shows curves obtained in solution of the coupled problem of the distributions of the dimensionless stresses $s = \sigma/E_2$ over the beam section. The dimensionless coordinate $\xi = z/h$ is plotted on the ordinate. Curve 1 corresponds to $t = -0.8$ (the transformation in the extreme fibers of the beam begins at $t = -0.84$), and curves 2–6 correspond to $t = -0.6$, -0.4, -0.2, 0, and 1, respectively. Evidently, the stress distribution over the section differs greatly from the linear distribution obtained in solution of the uncoupled problem. Because of the phase transition in the given case, as in the elastoplastic problem, the stress distribution over the beam section becomes more uniform compared with the distribution in the elastic problem. At $T > T^*$, the stresses on the convex side of the beam become compressive, and on the concave side of the beam, they are tensile. According to the results obtained, at negative and small positive values of t, the distribution of the martensite phase over the beam section differs from the uniform distribution typical of the solution of the coupled problem.

It should be noted that the front of completion of the transformation propagates over the beam section under cooling at a much higher velocity than the front of the beginning of the direct transformation. At $\beta h = 0.001$, the transformation in the extreme fibers is completed at $t = 0.9658$, and the front of completion of the transformation traverses the entire beam section as t changes from 0.9658 to 1. In general, the propagation of the fronts of beginning and completion of the transformation is a result of solution of the coupled problem. In the uncoupled problem with a uniform temperature distribution over the section, direct transformation begins simultaneously at all points of the section at $t = 0$ and is completed at $t = 1$.

3. Formulation of the Bending Problem in Increments. In what follows, we consider problems in which the curvature β is not fixed but is found by solution of the problem. In this case, it is not possible to obtain a solution in quadratures, and the problem has to be solved by steps. To obtain equations in increments, one should differentiate relation (1.7) and, on the right side of the resulting equation, replace $d\varepsilon^1$ and $d\varepsilon^2$ using (1.1) and (1.2), respectively:

$$z d\beta = \frac{d\sigma}{E} - \left[\frac{\sigma}{E^2}\frac{dE}{dq} - (B + a_0\varepsilon^2)\right] dq. \tag{3.1}$$

According to (1.3) and (1.5), we have

$$dq = \varphi[k \operatorname{sign}(\sigma)\, d\sigma - dT]\sqrt{1-q^2}, \tag{3.2}$$

where $\varphi = \pi/(2(M_1 - M_2))$ for the direct transformation and $\varphi = \pi/(2(A_1 - A_2))$ for the reverse transformation. Substituting (3.2) into (3.1) and solving this equation for $d\sigma$, we obtain

$$d\sigma = \frac{z\, d\beta + \psi\, dT}{1/E + k\psi \operatorname{sign}(\sigma)}, \quad \psi = \left(B + a_0\varepsilon^2 - \frac{1}{E^2}\frac{dE}{dq}\sigma\right)\varphi\sqrt{1-q^2}. \tag{3.3}$$

Relations (3.3) are valid in the transformation zone. In the elastic region on the right side of the first equation of (3.3), it is necessary to set $\psi = 0$.

Using (1.8) and (3.3), for the differential of the bending moment dM we have (in a rectangular beam)

$$dM = 2b\int_0^h d\sigma z\, dz = 2b(f_1\, d\beta + f_2\, dT), \tag{3.4}$$

where

$$f_1 = \int_{z_0}^{h} \frac{z^2\, dz}{1/E + k\psi \operatorname{sign}(\sigma)} + \frac{1}{3}E_2 z_0^3; \quad f_2 = \int_{z_0}^{h} \frac{\psi z\, dz}{1/E + k\psi \operatorname{sign}(\sigma)}.$$

4. Relaxation of the Extreme Layers of the Beam in the Direct Transformation under Pure Bending. Let the bending moment be fixed: $dM = 0$, and the temperature changes via the temperature interval of the direct or reverse transformation. In this case, from (3.4) we obtain the following relation for the curvature increment:

$$d\beta = -(f_1/f_2)\, dT. \tag{4.1}$$

The procedure for determining the curvature β that corresponds to a given (fixed) value of the bending moment and cooling from the completely austenite state to a given temperature reduces to the following. The curvature β^0 in the elastic problem and the corresponding stress distribution in the section $\sigma^0 = E_2\beta^0 z$ are determined. The temperature of the beginning of the transformation T^0 is obtained from formula (2.1).

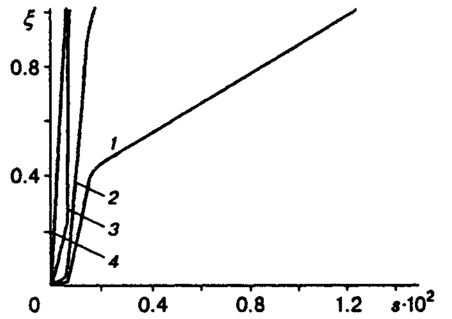

Fig. 3

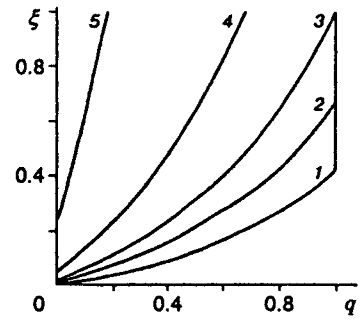

Fig. 4

The quantities T^0, β^0, and σ are used as initial values in step-by-step solution. The temperature increases with a chosen step dT, and the corresponding values of $d\beta$ and $d\sigma$ are determined using (3.3) and (4.1). The calculated increments are added to the values of β and σ determined in the previous step. After that, the quantity q for the chosen points of the section is found from (1.3), the elastic strain is found from (1.1), and the phase strain is obtained in the form $\varepsilon^2 = \beta z - \varepsilon^1$.

Using the above algorithm, we solved the problem of direct transformation of a beam of titanium nickelide under pure bending. It is established that at $T > M_1$, the stress distribution over the beam section is similar to that given in Fig. 2, which corresponds to the problem of stress relaxation in a beam of given curvature. On subsequent cooling to temperatures $T < M_1$, the pattern changes suddenly. The central part of the section relaxes so that the stresses in the region from the neutral curve to the relative height of the section, which is approximately equal to $\xi \cong 0.5$, are close to zero. At the same time, the extreme fibers of the beam are overstressed [for a relative value of the fixed bending moment $\mu = 3M/(2bh^2E_2) = 0.001$, the relative stresses in the extreme fibers for the elastic solution are also $s = 0.001$, and, in complete transformation, they are $s = 0.00188$, i.e., about twice the elastic stresses]. This phenomenon is related to the propagation of the front of completion of the transformation through the outer layers of the beam, after which the phase strain ceases to change and elastic deformation actually proceeds. In the problem of stress relaxation in the direct transformation in a beam with fixed curvature (see Fig. 2), overload of the outer layers was not observed, because the front of completion of the transformation travels very rapidly through the entire section at the very end of the cooling process.

5. Martensite Inelasticity in a Bent Beam. Let a beam in the austenite state be loaded by a monotonically increasing bending moment under isothermal conditions at $T > M_1$. This can lead to the occurrence of martensite inelasticity [1, 2] in the beam, in which case the direct transformation proceeds as a result of change in stress rather than temperature. A description of martensite inelasticity using constitutive equations (1.2) and (1.4) for a uniform stressed state in a stretched beam is given in [6]. For a bent beam, this phenomenon can be described using the above-stated algorithm.

It should be noted that martensite inelasticity is intimately related to the effect of the acting stresses on the phase composition and, hence, can be described only in solution of coupled problems, for which it is necessary to set $dT = 0$ in (3.4). As a result, the curvature increment is determined from the formula $d\beta = dM/(2bf_1)$.

The procedure for solution of the problem is similar to that described above. The only difference is that here in the initial stage at a given temperature T, we find the initial value of the moment $M_0 = (T - M_1)J/(kh)$ at which the direct transformation begins in the extreme fiber of the section considered and we use the results of elastic solution for $M = M_0$ as the initial values of curvature and stresses. Solution in increments is performed with a chosen step of the moment dM, beginning with M_0.

Figure 3 shows distribution curves of the relative stress $s = \sigma/E_2$ over the beam section under an isothermal increase in the moment. The calculation is performed at $t = -0.5$. Curves 1–4 correspond to loading to the relative bending moments $\mu = 0.01$, 0.002, 0.001, and 0.0006. Curve 4 corresponds to loading under which the transformation begins for the first time in the extreme fiber of the beam (therefore, the stress distribution over the section is linear, as is typical of the elastic problem). The point of inflection on curve 3 corresponds to the position of the front of the beginning of the transformation, and curve 2 corresponds to the maximum moment at which the transformation is first completed at the extreme fiber of the beam. At large values of the maximum moment, the curves of stress distribution over the section are three-segment broken lines with two straight segments (in the elastic zone containing the neutral curve and in the zone adjacent to the fibers, in which the transformation is already completed) and a curvilinear segment in the zone of continuing transformation located between the above-mentioned zones. The slope of the straight segments $d\sigma/dz$ in the zone of complete transformation increases with increase in the bending moment. The central part of the beam appears to be unloaded, and the extreme fibers are overstressed compared to the elastic solution.

Figure 4 shows distribution curves over the section for the portion of the martensite phase q. Curves 1-5 correspond to $\mu = 0.01$, 0.004, 0.002, 0.0015, and 0.001. In the problem considered, the fronts of both beginning and completion of the transformation move toward the neutral curve with increase in the bending moment. Here, however, at $T > M_2$, the front of the beginning of the transformation never reaches the neutral curve, tending to it asymptotically as the bending moment increases indefinitely.

If it is necessary to find the deflection of the beam W from the given distribution of the bending moment at small strain and deflection, the approximate expression for the curvature β (1.9) is represented in finite differences. As a result, using central finite differences, one obtains the following recursion relation for the deflection:

$$W(x_{i+2}) = \beta[M(x_{i+1}), T](\Delta x)^2 - W(x_i) + 2W(x_{i+1}), \quad i = 1, 2, \ldots, n-2. \tag{5.1}$$

Here the neutral axis of the beam is divided into equal segments of length Δx by the points $x_1, x_2, \ldots, x_n$, the values of $W(x_1)$ and $W(x_2)$ are determined from the conditions of attachment of the left end of the beam, and the quantities $\beta[M(x_{i+1}), T]$ are found by the procedure described above. This procedure is used to determine the deflections of a cantilever beam clamped at one end and loaded by a transverse force at the other end.

The problem was solved in two variants. In the first, the beam was cooled at a constant loading P to a certain temperature T. In the second variant, the load was increased from zero to P under isothermal conditions at $T \geqslant M_1$. It is established that in the case of martensite inelasticity, other conditions being equal (the same temperatures not exceeding M_1 and the same maximum loading values), values of curvature and deflection larger than in the direct transformation at $P = \text{const}$ are attained. At the same time, cooling the beam to a temperature $T = M_2$ under constant load, one can attain somewhat larger deflection values than those under the same maximum loading under isothermal conditions at $T = M_1$.

6. Solution of the Problem of Bending of a Beam with Unilateral Coupling at the Right End. Solution of statically indeterminate problems is more difficult. Below we consider a coupled problem of cooling of a beam whose left end is clamped and whose right end in the austenite state is imparted (by means of a support) a deflection W_0 that can increase but cannot decrease. The bending moment at the right end is equal to zero. If the deflection of the right end exceeds W_0 (the plate separates from the support), the transverse force at the right end also vanishes. This problem in an uncoupled formulation was solved in [8].

Under cooling, phase deformation results from the direct transformation, and the transverse force acting on the beam decreases. Therefore, the distribution of the bending moment is not known beforehand and must be obtained by solution of the problem. To avoid use of the fourth-order bending equation, which involves numerical differentiation of the curvature defined in increments, one can use the fact that the problem in deflection increments is linear. Therefore, for a given small increment of the temperature dT, one can calculate the increment of the force at the right end dP for which zero increment of the deflection of the right end corresponds to the pair (dT, dP):

$$dW(x_n) = 0. \tag{6.1}$$

For this, Eq. (5.1) is written in increments:

$$dW(x_{i+2}) = d\beta[M(x_{i+1}), T](\Delta x)^2 - dW(x_i) + 2dW(x_{i+1}). \tag{6.2}$$

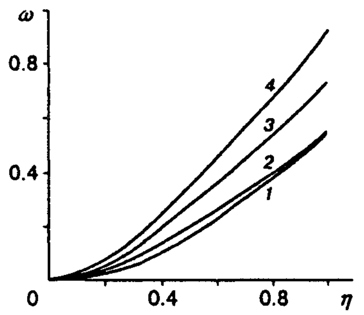

Fig. 5

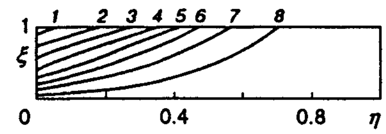

Fig. 6

The quantity $d\beta$ is determined from (3.4) in the form

$$d\beta(x_{i+1}) = A(x_{i+1})\,dP + B(x_{i+1})\,dT, \tag{6.3}$$

where

$$A(x_{i+1}) = \frac{1}{2b}\,\frac{L - x_{i+1}}{f_1}, \quad B(x_{i+1}) = -\frac{f_2}{f_1} \tag{6.4}$$

(L is the beam length). In (6.4), in calculation of the integrals entering the functions f_i according to (3.4), the quantity z_0 and also the functions in the integrands are taken for the section with the coordinate x_{i+1}. From the conditions of clamping of the left end, $dW(x_1) = dW(x_2) = 0$. Substituting (6.3) into (6.2) yields

$$dW(x_i) = (\Delta x)^2(D_i\,dP + F_i\,dT).$$

Here

$$D_i = \sum_{j=2}^{i-1}(i-j)A(x_j); \qquad F_i = \sum_{j=2}^{i-1}(i-j)B(x_j).$$

As a result, the absence of constancy for the deflection of the right end (6.1) can be written in the form

$$dP = -\frac{F_n}{D_n}\,dT. \tag{6.5}$$

Relation (6.5) yields the desired increment of the transverse force exerted on the beam by the support. Precisely this value is used for substitution into the right side of formula (6.3), which is used to determine the curvature increment. Further solution is performed by the procedure described above.

In solution of this problem, it is found that the force P exerted on the beam by the support decreases on cooling and reaches a zero value at a certain temperature T^*. On further cooling in the problem with bilateral attachment of the right end, the force P becomes negative and its absolute value increases. If the right end can move upward freely, on further cooling it separates from the support and its displacement increases with increase in temperature. This phenomenon was detected experimentally by Vitaikin et al. [12]. To describe the displacement of the beam under such conditions, in Eq. (6.3) at $T > T^*$, one should set $A(x_{i+1}) = 0$ for all i. Otherwise, solution is performed by the scheme described above.

Figure 5 shows curves of the position of the neutral curve of a beam of titanium nickelide at $W_0/h = 0.133$ and $L/h = 20$. The relative longitudinal coordinate $\eta = x/L$ is plotted on the abscissa, and the relative deflection $\omega = W/h$ is plotted on the ordinate. Curves 1 and 2 correspond to $t = -0.8$ and 0.1 (the right end has not separated from the support). The moment of separation corresponds to $t = t^* = 0.17$. Curves 3 and 4 are calculated for $t = 0.5$ and 1. Figure 6 illustrates the motion of the boundary of the zone of the beginning of the transformation along the same beam. For illustration, the scale of the beam height is increased fourfold. An elastic region is located below the corresponding curves, and the transformation zone is located above them. Curves 1–8 correspond to $t = -0.8$ -0.7, -0.6, -0.5, -0.4, -0.3, -0.2, and -0.1. At $t \geqslant 0$, the transformation zone occupies the entire beam.

In solution of the same problem in an uncoupled formulation, the relative temperature that corresponds to the moment of separation is determined from formulas (2.11) and (2.12), i.e., it coincides with the value of t^* in the problem of stress relaxation in a beam with a given curvature.

From experimental data, Vitaikin et al. [12] established that the maximum deflection of the beam end in complete direct transformation W is proportional to the initially given deflection W_0. Calculations show that, although the ratio W/W_0 is not constant, it changes insignificantly with variation in W_0 (when the value of W_0 changes by a factor of 3, the value of W/W_0 changes from 1.67 at $W_0/h = 0.06667$ to 1.66 at $W_0/h = 0.2$).

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